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DIFFERENTIAL TURNING POINT IN THE THEORY OF SINGULAR PERTURBATIONS. II

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Statement of problem

Let us consider the following singularly perturbed differential equation (SPDE)

$$\mathbf{L}_\varepsilon y(x,\varepsilon) \equiv \varepsilon^3 y'''(x,\varepsilon) + x\widetilde{a}(x)y'(x,\varepsilon) + b(x)y(x,\varepsilon) = h(x) \qquad (0.1)$$

as $\varepsilon \to +0$, $x \in I = [0,1]$.

The classical Liouville equation is a particular case of SPDE (0.1) for $b(x) \equiv 0$. Therefore, to the whole variety of problems related to the Liouville equation, for SPDE (0.1) some new peculiarities and difficulties of investigation are added. For example, for the Liouville equation the two basic cases are possible as concerns the sign of the coefficient $\widetilde{a}(x)$. If $\widetilde{a}(x) > 0$, then the roots of the characteristic equation are purely imaginary and the proper turning point is stable. However, if $\widetilde{a}(x) < 0$, then the roots of the characteristic equation are real. In this case, the point $x = 0$ is an instable turning point (one of the Airy functions increases unboundedly at infinity).

For SPDE (0.1) the structure of the solution of equation (0.1) depends already on the signs of the two coefficients $\widetilde{a}(x)$ and $b(x)$. Therefore we will distinguish the following two basic dispositions of these functions.

Case 1. Let $\widetilde{a}(x) > 0$, $b(x) < 0$ for $x \in I$. Then $x = 0$ is a stable turning point for SPDE (0.1) and the general solution of the degenerate equation

$$\mathbf{L}_0\omega(x) \equiv x\widetilde{a}(x)\omega'(x) + b(x)\omega(x) = h(x) \qquad (0.2)$$

is sufficiently smooth for all $x \in [0,1]$, containing an arbitrary constant of integration. This means that in order to construct one linearly independent solution of SPDE (0.1) one can use the solution of the degenerate equation (0.2). This case was studied by the author in [1].

Case 2. Let $\widetilde{a}(x) < 0$, $b(x) > 0$. In this case, $x = 0$ is already an instable turning point for SPDE (0.1). However, the solution of the degenerate equation is sufficiently smooth for all $x \in [0,1]$ and also contains an arbitrary constant of integration. Consequently, as in the first case, it can also be used for construction of a linearly independent solution of SPDE (0.1). This case, in comparison with the previous one, does not contribute to difficulties of the construction of the asymptotic of solution to SPDE (0.1). Taking into account the technique developed for the construction of uniformly applicable asymptotic of solution for algebraic turning points (see [2]–[5]) and the methods in [1], we can say that, basically, Case 2 represents only some technical difficulties.

Case 3. Let $\widetilde{a}(x) > 0$, $b(x) > 0$. Here, as in the first of cases, $x = 0$ is a stable turning point for SPDE (0.1). However, both the solution of the degenerate equation and its derivatives already are not sufficiently smooth at the point $x = 0$; more exactly, the solution of the degenerate equation has a discontinuity of the second kind at the turning point. Therefore, it cannot be used for the construction of a linearly independent solution of SPDE (0.1). This case already creates principal