

# Sufficient Conditions for Realizability of Boolean Functions by Asymptotically Optimal Circuits with the Unreliability $2\varepsilon$

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**Abstract**—We consider realizations of Boolean functions by circuits composed of unreliable functional elements in some complete finite basis  $B$ . We assume that all elements independently of each other with the probability  $\varepsilon$  ( $\varepsilon \in (0; 1/2)$ ) are subjected to inverse failures at the output.

We construct Boolean functions  $\varphi(x_1, x_2, x_3)$  such that the presence of at least one of them in the considered basis  $B$  guarantees the realizability of all three Boolean functions by circuits whose reliability does not exceed  $2\varepsilon + 144\varepsilon^2$  with  $\varepsilon \leq 1/960$ . In addition, if  $B \subset B_3 \setminus G$  ( $B_3$  is the set of all Boolean functions of three variables  $x_1, x_2, x_3$ , and  $G$  is the set of Boolean functions such that each one of them is congruent either to  $x_1^{\sigma_1}x_2^{\sigma_2} \vee x_1^{\sigma_1}x_3^{\sigma_3} \vee x_2^{\sigma_2}x_3^{\sigma_3}$ , or to  $x_1^{\sigma_1}x_2^{\sigma_2} \oplus x_3^{\sigma_3}$ , or to  $x_1^{\sigma_1}x_2^{\bar{\sigma}_2} \vee x_2^{\sigma_2}x_3^{\sigma_3}$  ( $\sigma_1, \sigma_2, \sigma_3 \in \{0, 1\}$ )), then the presence of at least one of functions  $\varphi(x_1, x_2, x_3)$  guarantees the realizability of almost all Boolean functions by asymptotically optimal (with respect to the reliability) circuits, whose unreliability equals  $2\varepsilon$  as  $\varepsilon \rightarrow 0$ .

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We consider the realization of Boolean functions by circuits composed of unreliable functional elements in an arbitrary complete finite basis  $B$ . We assume that all elements of a circuit are subject to inverse failures on the outputs independently of each other with the probability  $\varepsilon$  ( $\varepsilon \in (0; 1/2)$ ). These failures are characterized by the fact that in the fault-free state a functional element realizes an assigned Boolean function  $e$ , and in the failed state it does the function  $\bar{e}$ .

We assume that a circuit  $S$  composed of unreliable elements realizes a Boolean function  $f(x_1, \dots, x_n)$ , if with a collection  $\tilde{a} = (a_1, \dots, a_n)$  that appear in the circuit inputs, the output value equals  $f(\tilde{a})$ , provided that the circuit has no failures. We denote by  $P_{f(\tilde{a})}(S, \tilde{a})$  the probability of error of the input set  $\tilde{a}$  of the circuit  $S$  that realizes the function  $f$ . We call the number  $P(S) = \max_{\tilde{a}} P_{f(\tilde{a})}(S, \tilde{a})$  the unreliability of the circuit  $S$ . The reliability of the circuit  $S$  equals  $1 - P(S)$ .

Let  $P_\varepsilon(f) = \inf_S P(S)$ ; here  $\varepsilon$  is the probability of the inverse failure on the output of one element, and the infimum is taken over all circuits  $S$  composed of unreliable elements that realize the function  $f(x_1, x_2, \dots, x_n)$ . A circuit  $A$  composed of unreliable elements that realizes a function  $f$  is said to be asymptotically optimal with respect to the reliability, if  $P(A) \sim P_\varepsilon(f)$  as  $\varepsilon \rightarrow 0$ , i.e.,  $\lim_{\varepsilon \rightarrow 0} \frac{P_\varepsilon(f)}{P(A)} = 1$ .

In [1] and [2] one has extracted the set of functions  $G = G_1 \cup G_2 \cup G_3$ , depending on variables  $x_1, x_2, x_3$ , where  $G_1$  is the set of functions congruent to those  $x_1^{\sigma_1}x_2^{\sigma_2} \vee x_1^{\sigma_1}x_3^{\sigma_3} \vee x_2^{\sigma_2}x_3^{\sigma_3}$ ,  $G_2$  is the set of functions congruent to those  $x_1^{\sigma_1}x_2^{\sigma_2} \oplus x_3^{\sigma_3}$ , and  $G_3$  is the set of functions congruent to those  $x_1^{\sigma_1}x_2^{\bar{\sigma}_2} \vee x_2^{\sigma_2}x_3^{\sigma_3}$ , where  $\sigma_1, \sigma_2, \sigma_3 \in \{0, 1\}$ . The set  $G$  contains fifty six functions. In [3] one has found the set  $G_4$  of functions of variables  $x_1, x_2, x_3$ , and  $x_4$  that are congruent to functions  $x_1^{\sigma_1}x_2^{\sigma_2} \vee x_3^{\sigma_3}x_4^{\sigma_4}$  or

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