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THE CAUCHY FUNCTION OF A FUNCTIONAL-DIFFERENTIAL EQUATION IN A BANACH SPACE

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Let R^m be a finite-dimensional Euclidean space with norm $|\cdot|$; $L_p([a,b],R^m)$ the space of functions $y:[a,b]\to R^m$ summable in the p-th $(1\le p<\infty)$ power, $\|y\|_{L_p}=\Big(\int\limits_a^b |y(s)|^p ds\Big)^{1/p}$; $L_\infty([a,b],R^m)$ the space of measurable essentially bounded functions $y:[a,b]\to R^m$, $\|y\|_{L_\infty}=\operatorname{vrai}\sup\limits_{t\in[a,b]}|y(t)|$; $AC_p([a,b],R^m)$ the space of absolutely continuous functions $x:[a,b]\to R^m$, $1\le p\le\infty$, with derivative $x'\in L_p([a,b],R^m)$, $\|x\|_{AC_p}=\|x'\|_{L_p}+|x(a)|$. In the above notation, we will omit the indices $m=1$, $p=1$. Besides that, in the notation of functional spaces, we will not indicate the domains and ranges of functions from these spaces when it can be done without ambiguity.

Many classes of equations with respect to an unknown differentiable function of one variable $x(\cdot)$ can be represented as an equation $Fx=0$ with operator $F:AC_p([a,b],R^m)\to L_p([a,b],R^m)$. This equation, called functional-differential, was studied by the participants of the Perm seminar under direction of Prof. N.V. Azbelev [1]. In the 80's, at this seminar, the idea arose to study an abstract analogue of functional-differential equations with operators acting in arbitrary Banach spaces. In the past years, the theory of such equations was elaborated and particular interesting realizations of general results were found [1]–[4]. The theory united "classical" representatives of functional-differential equations with less developed singular equations, impulse systems, hybrid systems, etc.

In this paper, we develop the theory of linear abstract functional-differential equations with Volterra operators. Although the equations under study are of rather general type, they retain the specificity of equations with aftereffect and are natural generalization of these equations. The paper is devoted to the definition, properties and an approximate construction of the Cauchy function, which is, as in the case of "usual" functional-differential equations, the most important tool for the study of a linear abstract equation.

1. Boundary value problem. The Green function

Let D and B be Banach spaces, and let D be isomorphic and isometric to the direct product $B\times R^n$. The system of equations

$$\mathcal{L}x=f,\quad lx=\alpha, \tag{1}$$

where $\mathcal{L}:D\to B$, $l:D\to R^n$ are linear bounded operators, is called a *boundary value problem* [3], [4]. It is assumed that the operator $\mathcal{L}$ is Noetherian, $\operatorname{ind}\mathcal{L}=n$. If problem (1) has a unique solution for each pair $(f,\alpha)\in B\times R^n$, then this solution can be represented [3] in the form $x=Gf+X\alpha$.

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