

A GENERALIZATION OF THE DRESSLER THEOREM

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In 1970 Dressler obtained an elementary proof of the following bound for the Euler’s φ -function:

$$\sum_{\varphi(n) \leq x} 1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} x(1 + o(1)).$$

Nicolas, Balasard, and Smati refined this result by means of Bateman’s analytical method (see [1]). In the present article we generalize Dressler’s result for the case of an arbitrary multiplicative function without any use of the analytical method. Indeed, we prove the following

Theorem. *Let $f(n)$, $|f(n)| \leq 1$, be a multiplicative function. If the series*

$$\sum_p \frac{1 - f(p)}{p}$$

converges, then there is valid the bound

$$N_f(x) = \sum_{\varphi(n) \leq x} f(n) = A_f x(1 + o(1)),$$

where

$$A_f = \prod_p \left(1 + \frac{p}{p-1} \sum_{t=1}^{\infty} \frac{f(p^t)}{p^t}\right) \left(1 - \frac{1}{p}\right).$$

The proof of Theorem follows from the

Lemma. *Let $f(n)$ ($|f(n)| \leq 1$) be a multiplicative function. Then for any y such that $\log y > \sqrt{\log x}$ and $(\log x)/\log y \rightarrow \infty$ there holds the bound*

$$N_f(x) = \sum_{\varphi(n_1) \leq \sqrt{x}} f(n_1) \sum_{n_2 \leq \frac{x}{\varphi(n_1)}} f(n_2) + o(x),$$

where n_1 means natural numbers whose prime divisors do not exceed y , or $n_1 = 1$, and n_2 means natural numbers, whose prime divisors exceed y , or $n_2 = 1$.

Proof of Lemma. We have

$$N_f(x) = \sum_{\varphi(n_1) \leq \sqrt{x}} f(n_1) \sum_{\varphi(n_2) \leq \frac{x}{\varphi(n_1)}} f(n_2) + O\left(\sum_{\sqrt{x} < \varphi(n_1) \leq x} \sum_{\varphi(n_2) \leq \frac{x}{\varphi(n_1)}} 1\right). \quad (1)$$

Write $v = x \exp((\log x)/y \log 2 + O(1/y))$, then one can easily show that $\varphi(n_2) \leq x$ yields $n_2 \leq v$. Hence, the remainder in the right side of (1) is \ll

$$x \sum_{\sqrt{x} < \varphi(n_1) \leq x} \frac{1}{\varphi(n_1)} \sum_{n_2 \leq v} \frac{1}{\varphi(n_2)} \ll x \frac{\log x}{\log y} \sum_{\sqrt{x} < \varphi(n_1) \leq x} \frac{1}{\varphi(n_1)}.$$

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