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# A GENERALIZATION OF THE DRESSLER THEOREM

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In 1970 Dressler obtained an elementary proof of the following bound for the Euler's $\varphi$-function:

$$\sum_{\varphi(n)\le x} 1 = \frac{\zeta(2)\zeta(3)}{\zeta(6)} x(1+o(1)).$$

Nicolas, Balasard, and Smati refined this result by means of Bateman's analytical method (see [1]). In the present article we generalize Dressler's result for the case of an arbitrary multiplicative function without any use of the analytical method. Indeed, we prove the following

**Theorem.** *Let $f(n)$, $|f(n)|\le 1$, be a multiplicative function. If the series*

$$\sum_p \frac{1-f(p)}{p}$$

*converges, then there is valid the bound*

$$N_f(x) = \sum_{\varphi(n)\le x} f(n) = A_f x(1+o(1)),$$

*where*

$$A_f = \prod_p \Big(1 + \frac{p}{p-1}\sum_{t=1}^{\infty}\frac{f(p^t)}{p^t}\Big)\Big(1-\frac{1}{p}\Big).$$

The proof of Theorem follows from the

**Lemma.** *Let $f(n)$ ($|f(n)|\le 1$) be a multiplicative function. Then for any $y$ such that $\log y > \sqrt{\log x}$ and $(\log x)/\log y \to \infty$ there holds the bound*

$$N_f(x) = \sum_{\varphi(n_1)\le\sqrt{x}} f(n_1) \sum_{n_2\le \frac{x}{\varphi(n_1)}} f(n_2) + o(x),$$

*where $n_1$ means natural numbers whose prime divisors do not exceed $y$, or $n_1 = 1$, and $n_2$ means natural numbers, whose prime divisors exceed $y$, or $n_2 = 1$.*

**Proof of Lemma.** We have

$$N_f(x) = \sum_{\varphi(n_1)\le\sqrt{x}} f(n_1) \sum_{\varphi(n_2)\le \frac{x}{\varphi(n_1)}} f(n_2) + O\Big(\sum_{\sqrt{x}<\varphi(n_1)\le x}\ \sum_{\varphi(n_2)\le\frac{x}{\varphi(n_1)}} 1\Big). \quad (1)$$

Write $v = x\exp((\log x)/y \log 2 + O(1/y))$, then one can easily show that $\varphi(n_2)\le x$ yields $n_2\le v$. Hence, the remainder in the right side of (1) is $\ll$

$$x \sum_{\sqrt{x}<\varphi(n_1)\le x} \frac{1}{\varphi(n_1)} \sum_{n_2\le v}\frac{1}{\varphi(n_2)} \ll x\frac{\log x}{\log y}\sum_{\sqrt{x}<\varphi(n_1)\le x}\frac{1}{\varphi(n_1)}.$$

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