Russian Mathematics (Iz. VUZ) *Izvestiya VUZ. Matematika*
Vol. 44, No. 4, pp. 64–67, 2000 *UDC 517.988*

MINIMIZATION OF QUADRATIC FUNCTIONALS WITH CONSTRAINTS IN THE FORM OF EQUALITIES

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Wide classes of variational problems, which contain, for example, functionals with a deviating argument, are, as a rule, unsolvable by means of the classical variational calculus. Attempts to abandon these frameworks were made, e. g., in [1]. For solving such problems a method of reduction of variational problems to a problem of minimization of a functional in a certain Hilbert space was developed (see [2]–[5]). In what follows we suggest a general approach to such a reduction, based on the direct study of the extremal problem.

Let us recall that a linear selfadjoint bounded operator $U : \mathbf{D} \to \mathbf{D}$ acting in a Hilbert space $\mathbf{D}$ is said to be *positive definite* on the space $\mathbf{D}$ if $\langle Uy, y\rangle \geq 0$ for all $y \in \mathbf{D}$; *strictly positive definite* if $\langle Uy, y\rangle > 0$ for all $y \in \mathbf{D}$, $y \neq 0$; *strongly positive definite* if $\langle Uy, y\rangle \geq \gamma \|y\|^2$ with a certain $\gamma > 0$ and all $y \in \mathbf{D}$. The positive definiteness of the operator U is equivalent to the nonnegativeness of its spectrum. The strong positive definiteness of the operator U means that with a certain $\gamma > 0$ all points of the spectrum of the operator U are not less than γ and, in addition, the inverse operator exists U^{-1} (see [6], p. 249).

The known condition for resolvability of the problem of minimization of the quadratic functional

$$\frac{1}{2}\langle x, Ux\rangle - \langle x, f\rangle \longrightarrow \inf, \tag{1}$$

where $f \in \mathbf{D}$ (see, e. g., [2], [3]), will be formulated in the following form.

Theorem 1. *A point $x_0 \in \mathbf{D}$ is a solution of problem* (1) *if and only if $Ux_0 = f$ and the operator U is positive definite on the space $\mathbf{D}$. If the operator U is strictly positive definite on the space $\mathbf{D}$, then problem* (1) *has at most one solution. If the operator U is strongly positive definite on the space $\mathbf{D}$, then the operator U^{-1} is invertible and problem* (1) *has the unique solution $x_0 = U^{-1} f$.*

Consider the problem of minimization of a quadratic functional with linear constraints in the form of the equalities

$$\frac{1}{2}\langle x, Ux\rangle - \langle x, f\rangle \longrightarrow \inf, \quad \ell x = \alpha, \tag{2}$$

where the components of the linear vector functional $\ell : \mathbf{D} \to \mathbf{R}^k$ are linearly independent and $\alpha \in \mathbf{R}^k$.

We denote by $\mathbf{D}_\ell$ the kernel of vector functional ℓ and represent the space $\mathbf{D}$ as the direct sum $\mathbf{D} = \mathbf{D}_\ell \oplus \mathbf{D}^\ell$, where $\mathbf{D}^\ell$ is the space orthogonal to the space $\mathbf{D}_\ell$. Let $P : \mathbf{D} \to \mathbf{D}_\ell$ be an orthogonal projector onto the space $\mathbf{D}_\ell$, defined via the equality $P = I - \ell^*(\ell\ell^*)^{-1}\ell$ (I being the unit operator). Then $\overline{P} = I - P = \ell^*(\ell\ell^*)^{-1}\ell$ is the orthogonal projector onto the space $\mathbf{D}^\ell$.

Supported by Russian Foundation for Basic Research (99-01-01278, 96-15-96195) and Contest Center for Research in Fundamental Natural Sciences, St.-Petersburg.