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ON THE RESOLVABILITY OF VARIATIONAL INEQUALITIES WITH UNBOUNDED SEMIMONOTONE MAPS

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Let X be a uniformly convex Banach space, X^* its dual, which is assumed to be strictly convex, and $A : X \to 2^{X^*}$ a multivalued semimonotone operator with $D(A) = X$, i.e., the operator $T = A + C$ is monotone, where $C : X \to X^*$ is a strongly continuous map (see [1], p. 267). In addition, assume that $T : X \to 2^{X^*}$ is a maximal monotone operator.

Consider in X the variational inequality

$$\langle Ax - f, z - x\rangle \geq 0 \quad \forall z \in \Omega, \quad x \in \Omega, \tag{1}$$

where Ω is a closed convex set in X, and f is a fixed element in X^*.

An element $x \in \Omega$ will be called a solution of (1) provided that $y \in Ax$ can be found such that $\langle y - f, z - x\rangle \geq 0$ for all $z \in \Omega$.

Let $U : X \to X^*$ be a dual map in X. Under our assumptions U is a continuous bounded monotone coercive map (see [1], pp. 313, 322, 330). Construct an operator $Fx = U(x - Px)$, where $P : X \to \Omega$ is a projection in X onto Ω. Usually, F is called the penalty operator (see [2]). $F : X \to X^*$ is a single-valued bounded monotone map; besides, $Fx = 0$ if and only if $x \in \Omega$ (see [2]–[4]).

Definition (cf. [2], p. 190; [4], p. 96). A multivalued operator $A : X \to 2^{X^*}$ is said to be pseudomonotone on X if

a) for each $x \in X$, the set Ax is not empty, closed, and convex in X^*;

b) if $x_n \rightharpoonup x$, $y_n \in Ax_n$, and

$$\overline{\lim}\,\langle y_n, x_n - x\rangle \leq 0, \tag{2}$$

then, for each $y \in X$, an element $z(y) \in Ax$ can be found such that $\langle z, x - y\rangle \leq \underline{\lim}\,\langle y_n, x_n - y\rangle$.

Lemma. *Any semimonotone operator A ($D(A) = X$) is pseudomonotone on X.*

Proof. Let $x_n \rightharpoonup x$, $y_n \in Ax_n$, and (2) be valid. Then $\langle Cx_n, x_n - x\rangle \to 0$; hence,

$$\overline{\lim}\,\langle z_n, x_n - x\rangle \leq 0, \quad z_n = y_n + Cx_n \in Tx_n. \tag{3}$$

Since T is a maximal monotone operator, T is pseudomonotone in the sense of the definition in [4] (p. 106); therefore (3) gives us

$$\underline{\lim}\,\langle y_n, x_n - y\rangle + \lim\langle Cx_n, x_n - y\rangle \geq \langle z, x - y\rangle;$$

where $z = z(y) \in Tx$. Consequently,

$$\underline{\lim}\,\langle y_n, x_n - y\rangle \geq \langle u, x - y\rangle, \quad u = u(y) \in Ax. \quad \Box$$