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Projective Equivalence of Smooth Functions on the Plane

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Abstract—In this paper, we find differential invariants for the action of the projective group on the space of smooth functions on the plane and propose a classification of orbits of this action.

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Introduction. The study of actions of geometric groups is one of important problems in geometry, which has many applications in various fields of mathematics (including algebra, theory of invariants, differential equations and others).

In [1], a classification of smooth functions on the plane with respect to the action of the linear group $\mathrm{GL}(2,\mathbb{R})$ has been obtained. Then, in [2], this classification has been generalized to the case of the actions of the symplectic and orthogonal groups on smooth functions in multidimensional spaces. In [3], affine classifications of smooth and rational functions on the plane have been obtained.

The aim of this paper is to construct a *projective* classification of smooth functions on the plane, i.e., a classification of smooth functions with respect to the action of the projective group $\mathrm{PGL}(2,\mathbb{R}) := \mathrm{GL}(3,\mathbb{R})/\{\lambda E : \lambda \in \mathbb{R}^*\}$ by linear-fractional changes of coordinates:

$$x \mapsto \frac{a_{11}x + a_{12}y + a_{13}}{a_{31}x + a_{32}y + a_{33}}, \quad y \mapsto \frac{a_{21}x + a_{22}y + a_{23}}{a_{31}x + a_{32}y + a_{33}}.$$

We will use the technics of differential invariants. First of all, we recall basic definitions and notation (see [4] for details).

Let $J^k\mathbb{R}^2$ denote the space of k-jets of functions $f : \mathbb{R}^2 \to \mathbb{R}$ with canonical coordinates $(x, y, u, u_x, u_y, \ldots)$. The projective group $\mathrm{PGL}(2,\mathbb{R})$ acts on the base $J^0\mathbb{R}$ by projective changes of coordinates (x, y). This actions is prolonged canonically to actions on the k-jet spaces $J^k\mathbb{R}^2$ for all k.

Definition 1. A *differential invariant of order* $\leqslant k$ of the action of the group $\mathrm{PGL}(2,\mathbb{R})$ is a rational $\mathrm{PGL}(2,\mathbb{R})$-invariant function $I : J^k\mathbb{R}^2 \to \mathbb{R}$.

First we construct two independent differential invariants.

Differential invariants. Note that a function $U := u$ is a differential invariant of order 0.

To construct the second invariant, we use the following geometrical reasoning. Consider a curve on $\mathbb{R}^2$ given by an equation $\{f(x, y) = 0\}$. We may assume that locally this curve is the graph of a function $y = h(x)$.

Now we will consider the action of the projective group on such curves or, what is the same, on functions $y = h(x)$. For this action, there exists the differential invariant I of order 7, called the *projective curvature* [5], which is expressed in terms of the derivatives $h_i := h^{(i)}$ of h of order no greater than 7 as follows:

$$I = \big(162h_2^6h_5h_7 - 810h_2^5h_3h_4h_7 + 1134h_2^5h_3h_5h_6 - 756h_2^4h_3^2h_5^2 + 13230h_2^4h_3h_4^2h_5 - 2835h_2^5h_4h_5^2$$

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