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STRONGLY RESONANCE ELLIPTIC VARIATIONAL INEQUALITIES WITH DISCONTINUOUS NONLINEARITIES

V.N. Pavlenko and E.A. Chizh

1. Problem statement and the main result

Let Ω be a bounded area in $\mathbf{R}^n$, $n \geq 2$, whose boundary Γ belongs to the class $\mathbf{C}_{2,\alpha}$, $0 < \alpha \leq 1$ ([1], p. 23), $K = \{v \in \overset{\circ}{\mathbf{W}}{}_2^1(\Omega) \mid v(x) \geq \psi(x)$ almost everywhere on $\Omega\}$, where $\psi \in \mathbf{C}_2(\overline{\Omega})$, $\psi|_\Gamma \leq 0$.

The problem implies finding a function $u \in K$ which satisfies the inequality

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij}(x) u_{x_i} (v-u)_{x_j} dx + \int_\Omega (a_0(x) - \lambda_0) u(x)(v-u)(x) dx + \\ + \int_\Omega g(x,u)(v-u)(x) dx \geq 0 \quad \forall v \in K. \quad (1)$$

Here $Lu(x) \equiv -\sum\limits_{i,j=1}^{n} (a_{ij}(x)u_{x_i})_{x_j} + a_0(x)u(x)$ is a uniformly elliptic differential operator, $a_{ij} \in \mathbf{C}_{1,\alpha}(\overline{\Omega})$, $a_0 \in \mathbf{C}_{0,\alpha}(\overline{\Omega})$ and $a_0(x) \geq 0$ on Ω, λ_0 is the minimal eigenvalue of the operator L with the boundary condition $u|_\Gamma = 0$. The function $g : D \to \mathbf{R}$ ($D = \{(x,\xi) \in \Omega \times R \mid x \in \Omega$ and $\xi \geq \psi(x)\}$) is superimposedly measurable, i. e., for any Lebesgue measurable function $u(x)$ on Ω with values $u(x) \geq \psi(x)$ the function $g(x,u(x))$ is Lebesgue measurable on Ω. In addition, we assume that for almost all $x \in \Omega$ the function $g(x,\cdot)$ has on $[\psi(x),+\infty)$ only discontinuities of the first kind and it is continuous for $\xi = \psi(x)$, $g(x,\xi) \in [g_-(x,\xi), g_+(x,\xi)]$ for any $\xi \in [\psi(x),+\infty)$, $g_-(x,\xi) = \liminf\limits_{s\to\xi} g(x,s)$, $g_+(x,\xi) = \limsup\limits_{s\to\xi} g(x,s)$.

Definition 1. The nonlinearity $g : D \to \mathbf{R}$ with respect to the linear differential operator $Bu = Lu - \lambda_0 u$ satisfies A1-*condition* if there exists at most a denumerable family of surfaces $\{S_i,\ i \in I\}$, $S_i = \{(x,\xi) \in D \mid \xi = \psi_i(x)\}$, $\psi_i \in \mathbf{W}^2_{1,\mathrm{loc}}(\Omega)$, such that for almost all $x \in \Omega$ the inequality $g_+(x,\xi) \neq g_-(x,\xi)$ implies the existence of $i \in I$, for which $(x,\xi) \in S_i$ and $(B\psi_i(x) + g_-(x,\psi_i(x)))(B\psi_i(x) + g_+(x,\psi_i(x))) > 0$, or $B\psi_i(x) + g(x,\psi_i(x)) = 0$.

Noncoercive variational inequalities with discontinuous nonlinearities were investigated by many mathematicians. Mention the paper [2] which contains a number of relevant references. A new investigation method for such problems is proposed in this paper. Elliptic variational inequalities with discontinuous nonlinearities were investigated by one of the authors by the variational method in [3], [4] and by the method of monotone operators in [5]. However, with reference to inequalities with differential operators in the latter three papers, the order $2m$ of the differential operator is assumed to exceed the dimension of the space variable and so the differential operators of the second order are excluded. In [6], a problem with an obstacle is defined as an elliptic boundary value problem with discontinuous nonlinearity. This statement is equivalent to a variational inequality.