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ALGEBRAS OVER THE OPERAD OF FINITE LABELED GRAPHS

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This paper is a continuation of the investigations which were started in [1]. In [1], two operad structures were defined on the family of square matrices with entries from a set which, in concrete examples, most often is a monoid. One of these structures was studied in details in [1]. In this paper, we study the other structure. In the case when the square matrices which are elements of this operad are defined over the two-element Boolean algebra, some its suboperads can be interpreted as sets of adjacency matrices of (oriented and nonoriented) graphs of certain types. Rejecting the condition of absence of loops allows us to introduce on the obtained operads structures of Epi-operads in the sense of [2]. This makes it possible to find (up to a rational equivalence) varieties of algebras over these operads, namely, to find the systems of operations for algebras of these varieties and the corresponding collections of identities. The main result of the paper was announced in [3].

We will use definitions and notation of [2]. Let $G(n)$ be the set of symmetric $n\times n$-matrices with elements from the two-element Boolean algebra $\{0,1\}$. The adjacency matrices of graphs (possibly with loops) without multiple edges are elements of this set. We also define the component $G(0)$ consisting of the empty matrix, which corresponds to the graph without vertices. We denote it by Λ and the corresponding adjacency matrix by $A(\Lambda)$. In analogy with [1], we define the operation of composition $G(m)\times G(n_1)\times\cdots\times G(n_m)\to G(n_1+\cdots+n_m)$ as follows:

$$AB_1\dots B_m=\begin{pmatrix} a_{11}+B_1 & \overline{a}_{12} & \dots & \overline{a}_{1m}\\ \overline{a}_{21} & a_{22}+B_2 & \dots & \overline{a}_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ \overline{a}_{m1} & \overline{a}_{m2} & \dots & a_{mm}+B_m \end{pmatrix}, \tag{1}$$

where $\overline{a}_{ij}=a_{ij}J_{n_in_j}$, $J_{n_in_j}$ is the matrix of type $n_i\times n_j$ consisting of units, and $a_{ii}+B_i$ is the matrix obtained from B_i by adding a_{ii} to each element of B_i.

We also define the family $\operatorname{Dir}G=\{\operatorname{Dir}G(n)\mid n\ge 0\}$, where $\operatorname{Dir}G(n)$ is the set of not necessarily symmetric $n\times n$-matrices with elements from the two-element Boolean algebra $\{0,1\}$.

Theorem 1. *With respect to the introduced operation of composition, $G=\{G(n)\mid n\ge 1\}$ and $\operatorname{Dir}G$ are operads with unit element the matrix* (0).

Proof is a matter of direct verification based on (1) and on Lemma 1 and Theorem 1 of [1]. □

Remark 1. $G(0)$ contains only one element, the empty matrix, $|G(0)|=1$. If among the matrices $B_1,\dots,B_m$ there is $B_i\in G(0)$, then the following rule should be added to the definition of composition (1): from A the i-th block row and the i-th block column are deleted and then the composition is taken with the other B_j. In terms of the graph theory, the substitution of the graph $\Gamma_i=\Lambda$ in place of the i-th vertex Γ_0 leads to the removing of this i-th vertex and all the edges (in the case of an oriented graph, all the arcs) incident to it.

With the use of the proof of Theorem 2 of [1] one can prove the following