UDC 531.1

ANALYTICAL STUDY OF THE LIMITING PROPERTIES OF A SUSPENDED CABLE SYSTEM WITH A RIGID BEAM

M.N. Kirsanov[1]

1 *c216@ya.ru*; National Research University MPEI

Analytical model of a cable-stayed fan designed bridge is examined. The statically indeterminate cantilever beam suspended on k elastic cables is revealed. The law of distribution of forces in the cables is identified. The obtained dependence of solutions on the number of cables can identify the asymptotic properties of the deflection of the structure and determine the optimal ratio of the size of the system under which the deflection becomes minimum. The algorithm for solving recurrence equations, method of induction and the gamma function are used in the analytic transformations. The exact solutions are obtained by means of computer algebra system Maple.

Keywords: computer algebra system, Maple, suspended cable system with a rigid beam.
Kirsanov

Introduction. Statement of the problem

Static analysis of cable-stayed resilient systems with real properties, structure and size generally is not particularly difficult and famous [1, 2]. However, the practical engineer and designer is often faced with the problem of assessing their computing. The test calculations of such complex systems, such as cable systems, are needed for practical and theoretical studies [3]. Exact analytical evaluations of solutions are particularly difficult to obtain. In many cases it is either difficult or impossible. This paper proposes a method for obtaining analytical solutions, based on the method of induction [4], combined with the capabilities of modern systems of analytical calculations.

Consider a cable-stayed system (cable-stayed bridge, fan design) consisting of a rigid cantilever beam hinged on a fixed support which hangs on k cables (Fig. 1, $k = 3$). Definitely the rigid beam does not enough accurate simulate the real system. However, to achieve the objective we had to go to such simplification. Some justification for this model is the real-life design with a large beam rigidity.

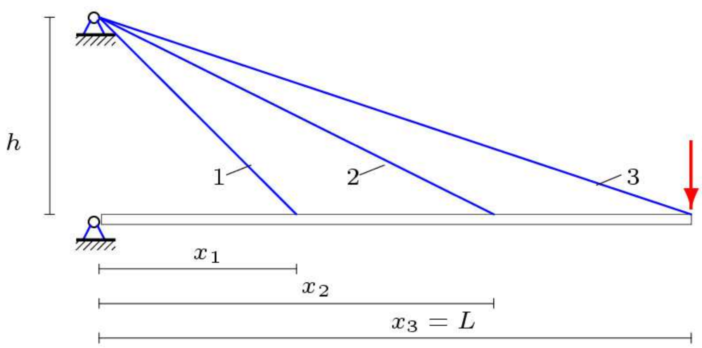

Fig. 1. The cable-stayed system, $k = 3$

The system is $k - 1$ times statically indeterminate. The vertical force P is attached to the beam. We number the cables from 1 to k. In the solution the length of the cables will included

$$l_j = \sqrt{h^2 + x_j^2}.$$

Coordinates x_j equally spaced at intervals of $a = L/k$ attachment points cables to the beam have the form

$$x_j = j\,a,\ \ j = 1,\ldots,k,$$

with $L = x_k$ — length of the console.

Determination of forces in cables

Static indeterminacy is disclose by the force method. As a primary determinate structure we will take a beam hanging on the k-th (far right) cable. Let's take cables stresses with the numbers $1,\ldots,k-1$ as redundant. Cable stiffness EF are same for all cables. The cables (except the k-th) are disconnected and the support of the cables $1,\ldots,k-1$ are replaced by the forces $X_j,\ j = 1,\ldots,k-1$ (Fig. 2).

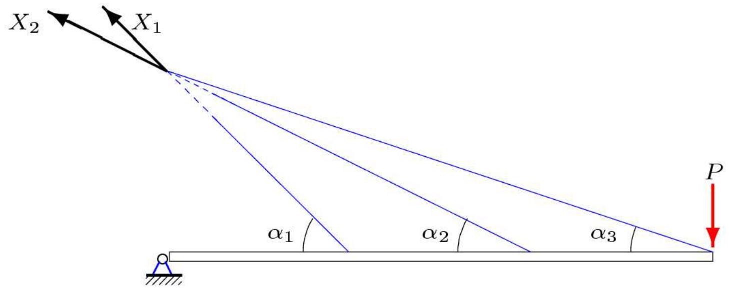

Fig. 2. the primary system 1

Let $S_{i,j}$ — be a force of a cable $i = 1,\ldots,k$ from the action of a unit force in the j-th cable, $j = 1,\ldots,k-1$. Forces $S_{i,k},\ i = 1,\ldots,k$ are calculated in the i-th cable per load P. Obviously, $S_{k,k} = P/\sin\alpha_k = Pl_k/h$. The forces in other cables, which are disconnected from the support in the core system are equal to zero $S_{i,k} = 0,\ i = 1,\ldots,k-1$. We have the following expressions

$$\begin{aligned} &S_{i,i} = 1,\ i = 1,\ldots,k-1,\\ &S_{i,j} = 0,\ i \neq j,\ i = 1,\ldots,k-1,\ j = 1,\ldots,k,\\ &S_{k,j} = -j\,l_k/(k\,l_j),\ j = 1,\ldots,k-1. \end{aligned}$$

Forces $S_{i,j}$ is the matrix which has a size $k \times k$. The lines corresponds to the numbers of cables, columns, $1,\ldots,k-1$ — numbers of unit stresses, the last column — stresses of cable of the load P. Obviously, the whole load is taken by the last right cable. For $k = 3$ the matrix takes the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\dfrac{1}{3}\dfrac{l_3}{l_1} & -\dfrac{2}{3}\dfrac{l_3}{l_2} & \dfrac{P\,l_3}{h} \end{bmatrix}. \tag{1}$$

The coefficients of the compatibility equations are computed as well.

$$\sum_{j=1}^{k-1} \delta_{i,j} X_j + \delta_{i,p} = 0, \quad i = 1, ..., k-1. \tag{2}$$

According to Maxwell's-Mohr formula we obtain

$$\begin{aligned} &EF\delta_{i,j} = \sum_{m=1}^{k} S_{m,i} S_{m,j} l_m = \frac{i\, j\, l_k^3}{k^2\, l_i\, l_j}, \\ &i \neq j, \; i, j = 1, ..., k-1, \\ &EF\delta_{i,i} = \frac{i^2\, l_k^3}{k^2\, l_i^2} + l_i, \\ &EF\delta_{i,p} = -P \frac{i\, l_k^3}{k\, l_i\, h}. \end{aligned} \tag{3}$$

In matrix form, for $k = 3$ the system has the form

$$\begin{bmatrix} \dfrac{{l_3}^3}{9{l_1}^2} + l_1 & \dfrac{2{l_3}^3}{9 l_1\, l_2} \\ \dfrac{2{l_3}^3}{9 l_1\, l_2} & \dfrac{4{l_3}^3}{9{l_2}^2} + l_2 \end{bmatrix} \begin{bmatrix} X_1 \\ \\ X_2 \end{bmatrix} + \begin{bmatrix} -\dfrac{{l_3}^3 P}{3 l_1\, h} \\ -\dfrac{2{l_3}^3 P}{3 l_2\, h} \end{bmatrix} = 0.$$

By induction [4] we find the solution of the system $k-1$ of the compatibility equations (2)

$$X_j = P\, k\, j\, l_j\, \beta_{j,k}/D, \; j = 1, ..., k-1, \tag{4}$$

where $\beta_{j,k} = \prod_{i=1}^{k} l_i^3 / l_j^3$ and D — determinant of the system

$$D = h \sum_{j=1}^{k} j^2\, \beta_{j,k}. \tag{5}$$

The force in the k-th (right) cable

$$X_k = \sum_{i=1}^{k-1} S_{k,i}\, X_i + S_{k,k} = P\, k^2\, l_k\, \beta_{k,k}/D.$$

The formal substitution $j = k$ in the expression (4), obtained strictly for $j = 1, ..., k-1$ gives the same value. This allows to continue using the solution (4) for all the cables $j = 1, ..., k$.

Another primary determinate structure

The best way to check for a solution of statically indeterminate system is to choose a different primary determinate structure.

We introduce a rigid beam with joints of the points of the attached cables, divided by k hinged parts.

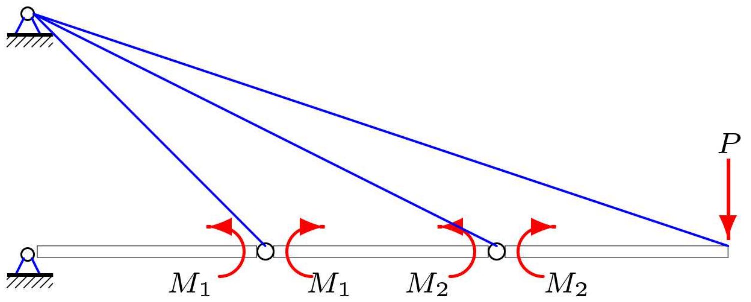

Fig. 3. the primary system 2

This system becomes statically determinate. Take it for the primary system of the method of forces. The moments M_i, $i = 1, \ldots, k-1$, introduced in the joints will be unknown of this method (Fig. 3).

To determine the forces $S_{i,\xi}$, $i = 1, \ldots, k$ in the rigging of the actions unit moments, corresponding unknowns M_ξ, $\xi = 1, \ldots, k-1$, we obtain the system of k equations. Each equation is the sum of the moments around the hinge with the coordinate x_{k-j}, $j = 1, \ldots, k$, (the moment point with the number j) of all forces that are attached to the severed right on the hinge side of the system. Therefore, moving from right to left on the beam, considering every time the balance of the right part from the hinge of the beam. The point has the number k — fixed support beams. The system of equations for the forces of the cables of a single moment of action which is in accordance with an unknown number ξ, is given by

$$\begin{aligned} &\Phi_j = 0, \ j = 1, \ldots, k, \ \ j \neq k-\xi, \\ &\Phi_{k-\xi} = 1. \end{aligned} \tag{6}$$

To refer to the sum of the moments we have introduced an auxiliary function

$$\Phi_j = \sum_{i=1}^{j} i\, a S_{i+k-j,\xi} \sin\alpha_{i+k-j}.$$

To calculate the forces from the actions of all the individual moments it is need to consistently make $k-1$ equations (6), respectively, for $\xi = 1, \ldots, k-1$.

Forces in the cables of the unit moments and loads of P are listed in the matrix similar to (1). Its non-zero elements are given by

$$\begin{aligned} &S_{i,i} = -2/(a\sin\alpha_i), \\ &S_{i+1,i} = 1/(a\sin\alpha_{i+1}), \ i = 1, \ldots, k-1, \\ &S_{i,i+1} = 1/(a\sin\alpha_i), \ i = 1, \ldots, k-2. \end{aligned}$$

Forces due to the load action P in all cables, except the far right cable, appended at the end of the beam are zero. In the k-th cable the force $S_{k,k} = P/\sin\alpha_k$.

For $k = 3$, we have a matrix of forces (the last column corresponds to the forces due to the load P)

$$\begin{bmatrix} -\dfrac{2l_1}{ah} & \dfrac{l_1}{ah} & 0 \\ \dfrac{l_2}{ah} & -\dfrac{2l_2}{ah} & 0 \\ 0 & \dfrac{l_3}{ah} & \dfrac{Pl_3}{h} \end{bmatrix}.$$

The non-zero coefficients of the compatibility equations for the unknown moments M_j, $j=1,...,k-1$,

$$\sum_{j=1}^{k-1}\delta_{i,j}M_j+\delta_{i,p}=0,\quad i=1,...,k-1, \tag{7}$$

that calculated similar to (3) are calculated according to the formula Maxwell's-Mohr, and have the form

$$\begin{aligned}
&\delta_{i,i}=\frac{\varphi_{i-1}+4\varphi_i+\varphi_{i+1}}{EF},\ i=1,...,k-1,\\
&\delta_{i,i+1}=-2\frac{\varphi_i+\varphi_{i+1}}{EF},\ i=1,...,k-2,\\
&\delta_{i,i+2}=\frac{\varphi_{i+1}}{EF},\ i=1,...,k-3,\\
&\delta_{i,j}=\delta_{j,i},\ i,j=1,...,k-1,\\
&\delta_{k-1,p}=\frac{Pa\varphi_k}{EF},
\end{aligned} \tag{8}$$

For brevity we introduce the shortcut $\varphi_i=l_i^3/(a^2h^2)$, $i=1,...,k$, $\varphi_0=0$. A five-diagonal matrix system is obtained. The determinant of the system (7) coincides with (5). The system of equations gives the values of the moments in the beam at the attachment points of cables and has the form

$$M_i=-ahPl_k^3\sum_{j=1}^{k-1}j\beta_{j,k-1}c_{i,j}/D,\quad i=1,...,k-1,$$

where the auxiliary matrix of coefficients $c_{i,j}$ is symmetric respect to both diagonals

$$\begin{aligned}
&c_{i,j}=(k-j)\,i,\ i=1,...,[(k-1)/2],\ j=i,...,k-i,\\
&c_{k-j,k-i}=c_{i,j}=c_{j,i},\ i=1,...,k-1,\ j=i,...,k-1,
\end{aligned}$$

The brackets $[(k-1)/2]$ denote round up. If $k=5$ the matrix c have the form

$$c=\begin{pmatrix}4&3&2&1\\3&6&4&2\\2&4&6&3\\1&2&3&4\end{pmatrix}.$$

We note two interesting properties of the matrix. First, the determinant of c of order n is equal to $(n+1)^{n-1}$ [17]. Obviously, $n=k-1$ is the indeterminate degree of structure. Second, the inverse matrix of c of order n, is a three-diagonal Jacobi's matrix with elements $2/(n+1)$ on the main diagonal and $-1/(n+1)$ on the other two. These properties can be used to monitor the accuracy of analytical transformations.

The forces in cables can be determined by formula

$$X_j=\sum_{i=1}^{k-1}S_{j,i}M_i+S_{j,k}.$$

The result in this case is exactly the solution (4). Note that selected for testing primary determinate structure with unknown moments in the joints, usually has used in the problem of multispan beam [6] (three-moments equation), in this case complicates but does not simplify solution.

Analysis of forces in cables

Dimensionless force values X_j/P in the last three cables $j = k-2, k-1, k$, depending on the height h, for $k = 12$, $L = 30\ m$ are given by Figure 4.

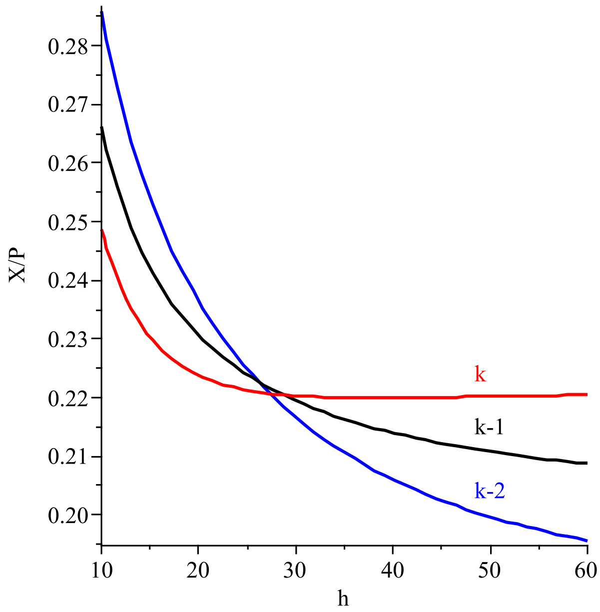

Fig. 4. Dimensionless force values in the last three cables

Height h in the Figure is given in meters. By increasing the height of the construction the stresses decrease in all rigging, though not always monotonic. On some curves one can see a small extremum. In this case — it's the last cable. Starting from a certain value h, it's force begins to rise. With an increasing number of cables k, calculations show that the cable number in which there is such a fluctuation is reduced. However, because of the small deviations from the change of stresses decreases monotonically with increasing h, to investigate this effect does not make sense.

It is much more important to explore the obvious asymptote. Moreover, this can be done analytically. If $h \to \infty$ expression (4), as it turns out, has a limit

$$\lim_{h\to\infty} X_j = \frac{6jP}{(k+1)(2k+1)},\quad j = 1, \dots, k. \tag{9}$$

This value is independent of the beam length L and decreases with the number of cables k. The linear dependence of the stress limit of cables on the number of j one could guess, if we solve the problem not by force method, but by the deformations method. In this case, at high altitudes h cables are almost vertical, and their extension is proportional to the distance from the support, ie, proportional to the number j. Hence, in view of Hooke's law it should be proportional to the force. Way to get the same exact dependence (9) is not obvious and deserves a more detailed explanation. The inductive method was used for the conclusions. If the number of cables k is given, then the dependence of (4) for large values of k is, though complicated, but it is a certain kind. Going to the limit by $h \to \infty$ is absolutely not difficult, and the result is simple. For example, if $k = 3$, we have $\lim\limits_{h\to\infty} X_j = 3jP/14 = 6jP/28$, for $k = 4$ we get $\lim\limits_{h\to\infty} X_j = 2jP/15 = 6jP/45$, etc. Thus, noting the numerator is 6, write down the sequence denominators 6, 15, 28, 45, 66, 91

... if $k = 1, 2, 3, 4, 5, 6, \ldots$. With the help of the operator rgf_findrecur from genfunc package of computer mathematics Maple [7, 8] we obtain the recurrence equation for the sequence of coefficients of the denominators

$$t_k = 3t_{k-1} - 3t_{k-2} + t_{k-3}, \; k \geq 4.$$

The operator rgf_findrecur is needed to work with even number of members of the analyzed series.

In this case, the detection of this equation has proved sufficient members of the three pairs of sequences (sometimes required more). The integer coefficients as a result is a sign that the equation are matched correctly. Indeed, all t_k must be integer. The solution of the recurrence equation (here came the third-order equation) can be easily found by using the rsolve or manually, by known methods of discrete mathematics. Obtain the necessary dependence for the denominator: $t_k = (k+1)(2k+1)$.

Distribution of relative stresses X_j/P the cables $j = 1, \ldots, 12$, depending on the height h (in meters) for $k = 12$, $L = 30$ m is given in Fig. 5.

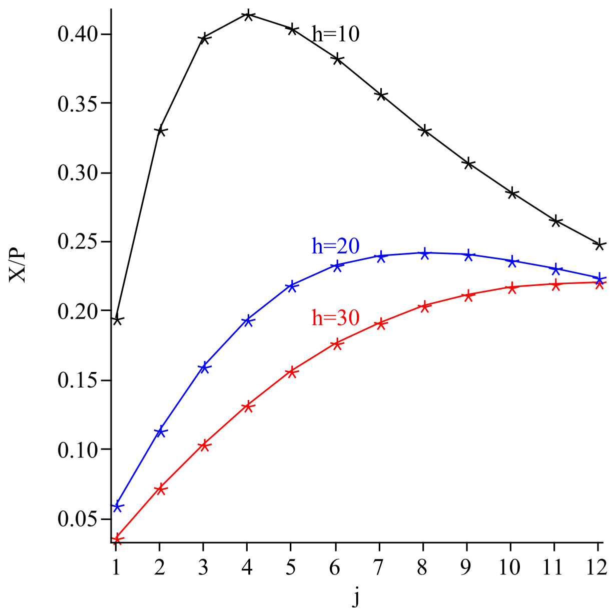

Fig. 5. Distribution of relative stresses the cables

There is the number of the cable with the maximum force among all the cables depends on h. This dependence can be found out analytically by differentiating (4) on j, considering this relationship to be a continuous function. We have

$$j_{\max} = kh/L.$$

Consequently, the extreme right-hand cable ($j_{\max} = k$) is strained more than others, if $h = L$, ie, it is inclined at an angle of 45°.

Analysis of deflection

The deflection of the console calculated by Maxwell's-Mohr formula

$$\Delta = \sum_{i=1}^{k} \bar{S}_{i,k} X_i l_i/(EF),$$

where $\bar{S}_{i,k} = S_{i,k}/P$ — forces in cables in the basic statically determinate system from the action of a unit force at the end of the beam. Given that in this case the forces of all cables, except the last one with the number k are zero and $\bar{S}_{k,k} = 1/\sin\alpha_k = l_k/h$, we get

$$\Delta = \frac{S_{k,k} X_k l_k}{EF} = \frac{P l_k^3 k^2 \beta_{k,k}}{hDEF}. \quad (10)$$

Let us analyze the deflection as a function of the number of cables. We denote $\tilde{\Delta} = EF\Delta/(PL)$ — dimensionless deflection. Increasing the number of cables, the rigidity of the whole structure will obviously rise and deflection at $k \to \infty$ will tend to zero. Another thing, if fighting for material savings, with the number of cables we will reduce their section. Thus it is possible to determine the dependence of the stiffness of the number of cables. Suppose, for example,

$$EF = EF_0 l_k / L_S,$$

where $L_S = \sum_{i=1}^{k} l_i$ — the total length of the cables, EF_0 — stiffness cables in the statically determinate system with one cable. With an increasing number of cables the total length increased as well. The length of the far right cable $l_k = \sqrt{L^2 + h^2}$ is constant, therefore, the cables stiffness decreases. The volume of cable material with a fixed modulus of elasticity remains constant $F_0 l_k$.

The Figure 6 shows the dependence of the dimensionless deflection of the number of cables at different altitudes $h = 10$ m, $h = 20$ m, $h = 40$ m, $L = 30$ m.

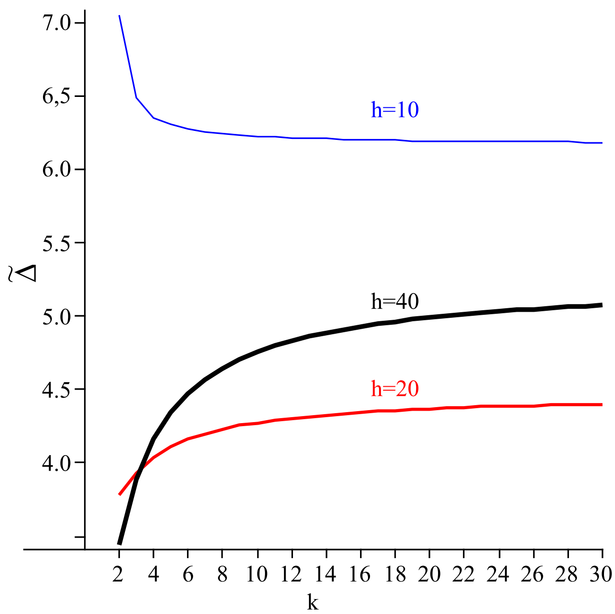

Fig. 6. Dependence of the dimensionless deflection of the number of cables

Obviously the presence of the asymptote of this relationship when $k \to \infty$.

For the analytical determination of the asymptote it is better to express the finite product in $\beta_{i,j}$ as a part of the solution (4) in terms of the gamma function [9, 10]

$$\prod_{j=1}^{k} h^2 + j^2 a^2 = a^{2k} \frac{\Gamma(k+1-ih/a)\Gamma(k+1+ih/a)}{\Gamma(1-ih/a)\Gamma(1+ih/a)},$$

here i — the imaginary unit.

Passing to the limit $k \to \infty$ in the expression (10), we obtain

$$\tilde{\Delta}_{\max} = \frac{\lambda}{2}\frac{\lambda\sqrt{1+\lambda^2} + \operatorname{arcsinh}\lambda}{\sqrt{1+\lambda^2}\operatorname{arcsinh}\lambda - \lambda}, \tag{11}$$

where the notation $\lambda = L/h$.

Similarly, in the simple way if the stiffness EF is divided to the number of cables ie, $EF = EF_0/k$, we can obtain

$$\tilde{\Delta}_{\max} = \frac{\lambda^2}{\operatorname{arcsinh}\lambda - \lambda/\sqrt{1+\lambda^2}}. \tag{12}$$

In this case, of course, the volume of the system as the function of number of cables is not saved, the solution turns out to be a little bit easier. When the ratio of the beam length and height of the structure $\lambda = 1.354$ limit (11) reaches a minimum value of $\tilde{\Delta}_{\max} = 4.448$, and the asymptote is calculated using the formula (12) has a minimum value of $\tilde{\Delta}_{\max} = 5.736$ at $\lambda = 1.027$. These numbers do not depend on the size of the system, nor on its elastic properties and are peculiar character of "universal constants" of problem, which can be guided in the process of designing such structures.

The conclusions

In the problem of statically indeterminate cabling system with an arbitrary number of cables, the exact solutions for the deflection and stress in cables has been found. The analytic dependence of the solution on the number of cables helped to identify some asymptotic properties of deformability of the structure. In process of solutions specific mathematical patterns allowing to move the proposed methodology inductive to other computing tasks has also been found.

Some other examples of application of the method of induction in conjunction with the system of computer mathematics Maple can be found in the works [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

Appendix. Bissymmetric matrix. Examples

Let us consider some bisymmetric matrix A of order n, i.e. symmetric about the main diagonal

$$a_{i,j} = a_{j,i}, \ \ i,j = 1,...,n$$

and on the side one

$$a_{n-j+1,n-i+1} = a_{j,i}.$$

1) $a_{i,j} = i + j, \ i = 1,...,[n/2], \ j = i,...,n+1-i$, For $n = 5$ the matrix has the form

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 5 \\ 4 & 5 & 6 & 5 & 4 \\ 5 & 6 & 5 & 4 & 3 \\ 6 & 5 & 4 & 3 & 2 \end{pmatrix}. \tag{A1}$$

Using the method of induction we have

$$\det A = 2^{n-2}(3+n)(\cos(\pi n/2) + \sin(\pi n/2)).$$

The matrix (A1) in the system Maple can be set as the Hankel matrix [8] with symmetric list:

HankelMatrix(<2,3,4,5,6,5,4,3,2>,5).

2) $a_{i,j} = j - i,\ i = 1,\dots,[n/2],\ j = i,\dots,n+1-i$, For $n = 5$ the matrix has the form

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}. \qquad (A2)$$

Using the method of induction we have

$$\det A = -2^{n-2}(n-2)(n-1).$$

Note that the matrix (A2) in the system Maple can be set as a BandMatrix [8].

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