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ON A CLASS OF DIFFERENCE METHODS FOR SOLVING NAVIER–STOKES EQUATIONS

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The present article is devoted to questions of construction of difference schemes and iterative methods for solving (u, v, P)-systems of viscous incompressible liquid. The main attention is paid to solving the linearized Navier–Stokes equations with use of the idea of multicomponent alternating direction method (MADM) (see [1]–[4]), which is close to methods of splitting (see [5], [7]) and possesses complete approximation (see [8], [9]). In the capacity of basic difference schemes, energetically neutral schemes are used, suggested for the Navier–Stokes equations in [10]. The choice of schemes of this form can be explained by the possibility to carry out computing without statement of additional conditions for the pressure on a boundary or in a domain which is inactive in the basic computation (which, in general, seems to be natural) (see [11], [12]) with exact fulfillment of the sticking conditions and difference equation of continuity. These conditions are hardly kept in the construction of economic methods. This concerns both the splitting method (see [6], [7]) and the method of alternating directions (see [13]). In this connection we can mention the works [14], [15], where on scattered grids an explicit scheme of splitting by physical factors is considered, allowing to satisfy the sticking conditions with high accuracy.

Energetically neutral difference schemes represent implicit algorithms of rather complicated construction, therefore to realize them one should construct economic methods which could preserve their properties of conservativeness and complete approximation.

In the present article we suggest economic methods of complete approximation for splitting equations with respect to both physical processes and dimension. We also suggest methods of decomposition of domain (see [16]). We study both iteration-free methods for nonstationary problems and iterative processes for solving both stationary and nonstationary equations of the dynamics of a viscous incompressible liquid. A considerable attention will be paid to the questions of parallelizing the computation algorithm (see [17], [18]).

In the domain $G_T = \overline{G} \times [0, T]$, $\overline{G} = \{x = (x_1, x_2),\ 0 \le x_\alpha \le l_\alpha,\ \alpha = 1, 2\}$, which is a rectangle with the boundary Γ, we consider a linearized two-dimensional boundary value problem for the Navier–Stokes equations in the velocity–pressure variables

$$\frac{\partial u_k}{\partial t} - \nu \sum_{\alpha=1}^{2} \frac{\partial^2 u_k}{\partial x_\alpha^2} + \sum_{\alpha=1}^{2} a_\alpha \frac{\partial u_k}{\partial x_\alpha} = -\frac{\partial P}{\partial x_k}, \quad (x, t) \in G \times (0, T], \quad \nu > 0, \quad k = 1, 2, \tag{1}$$

$$\sum_{\alpha=1}^{2} \frac{\partial u_\alpha}{\partial x_\alpha} = 0, \quad (x, t) \in G \times (0, T], \tag{2}$$

$$u_k(x, 0) = \varphi_k(x), \quad x \in \overline{G}, \quad u_k(x, t) = 0, \quad (x, t) \in \Gamma \times (0, T]. \tag{3}$$

In equation (1) the coefficients a_α, $\alpha = 1, 2$, are some bounded constants. In the statement thus given, from [19] the existence and uniqueness of problem (1)–(3) follow (the pressure P is determined

up to an additive constant).

To construct a numerical algorithm, in the domain G_T we introduce two spatial grids uniform in each direction: the grid of integer nodes $\overline{\omega}_h = \{x = (x_1^{(i_1)}, x_2^{(i_2)}) = (i_1 h_1, i_2 h_2),\ i_\alpha = 0, \dots, N_\alpha,\ h_\alpha = l_\alpha/N_\alpha,\ \alpha = 1, 2\}$ with boundary $\gamma_h = \overline{\omega}_h \setminus \omega_h$ and the grid of semi-integer nodes $\Omega_h = \{x = (x_1^{(i_1-0.5)}, x_2^{(i_2-0.5)}) = ((i_1-0.5)h_1, (i_2-0.5)h_2),\ i_\alpha = 1, \dots, N_\alpha,\ h_\alpha = l_\alpha/N_\alpha,\ \alpha = 1, 2\}$; a temporary grid is defined in a standard way $\overline{\omega}_\tau = \{t_n = n\tau,\ n = 0, 1, \dots\}$. In addition, we divide Ω_h into two grid subsets: $\Omega_h = S_h \cup T_h$, $S_h = \{x = (x_1^{(i_1-0.5)}, x_2^{(i_2-0.5)}),\ i_\alpha = 2, \dots, N_\alpha - 1,\ \alpha = 1, 2\}$, $T_h = \{x = (x_1^{(i_1-0.5)}, x_2^{(i_2-0.5)}),\ i_1 = 1, N_1,\ i_2 = 1, \dots, N_2;\ i_2 = 1, N_2,\ i_1 = 2, \dots, N_2 - 1\}$.

Let H_1 be a space of grid functions defined on $\overline{\omega}_h$ and vanishing on γ_h, with the scalar product $(u, v)_{H_1} = \sum\limits_{x \in \omega_h} h_1 h_2 u(x) v(x)$ and the norm $\|u\|_1 = (u, u)_{H_1}^{1/2}$. In a similar way, we define a space H_2 of grid functions given on the grid Ω_h, with the scalar product $[u, v]_{H_2} = \sum\limits_{x \in \Omega_h} \overline{h}_1 \overline{h}_2 u(x) v(x)$ and the norm $\|u\|_2 = [u, u]_{H_2}^{1/2}$, where $\overline{h}_\alpha = \begin{cases} h_\alpha, & i_\alpha = 2, \dots, N_\alpha - 1; \\ 0.5 h_\alpha, & i_\alpha = 1, N_\alpha, \quad \alpha = 1, 2. \end{cases}$ Further in H_1 we shall consider grid analogs of the velocities u_1, u_2, while in H_2 — grid analogs of the pressure P.

In addition to the notation which is usually adopted in the theory of difference schemes we shall write (for the sake of brevity, instead of i_1, i_2 we write i, j):

$$
\begin{aligned}
u_{\widehat{x}_1}^{i,j} &= (u_{i+1/2,\ j+1/2} + u_{i+1/2,\ j-1/2} - u_{i-1/2,\ j+1/2} - u_{i-1/2,\ j-1/2})/(2h_1), \quad u \in \Omega_h, \\
u_{\widehat{x}_2}^{i,j} &= (u_{i+1/2,\ j+1/2} + u_{i-1/2,\ j+1/2} - u_{i+1/2,\ j-1/2} - u_{i-1/2,\ j-1/2})/(2h_2), \quad u \in \Omega_h, \\
v_{\widetilde{x}_1}^{i-1/2,\ j-1/2} &= (v_{i,j} + v_{i,\ j-1} - v_{i-1,\ j} - v_{i-1,\ j-1})/(2h_1), \quad v \in \omega_h, \\
v_{\widetilde{x}_2}^{i-1/2,\ j-1/2} &= (v_{i,j} + v_{i-1,\ j} - v_{i,\ j-1} - v_{i-1,\ j-1})/(2h_2), \quad v \in \omega_h.
\end{aligned}
$$

By replacing the functions u_1, u_2, and P with their grid analogs y_1, $y_2 \approx u_1$, z_1, $z_2 \approx u_2$, and $q \approx P$, we approximate problem (1)–(3) by the following difference scheme of MADM:

$$
\begin{gathered}
Y_{1t} = -\operatorname{grad}_h \widehat{Q} + LY_2, \quad (x, t) \in \omega_h \times \omega_\tau, \\
Y_{2t} = -\operatorname{grad}_h \widehat{Q} + L\widehat{Y}_2, \quad (x, t) \in \omega_h \times \omega_\tau, \\
\operatorname{div}_h \widehat{Y}_1 \equiv \widehat{y}_{1\widetilde{x}_1} + \widehat{z}_{1\widetilde{x}_2} = 0, \quad (x, t) \in \Omega_h \times \omega_\tau, \\
y_\alpha(x, 0) = \varphi_1(x), \quad z_\alpha(x, 0) = \varphi_2(x), \quad x \in \overline{\omega}_h, \quad \alpha = 1, 2, \\
y_\alpha(x, t) = z_\alpha(x, t) = 0, \quad (x, t) \in \gamma_h \times \omega_\tau, \quad \alpha = 1, 2,
\end{gathered}
\tag{4}
$$

where $Y_\alpha = (y_\alpha, z_\alpha)^*$, $\alpha = 1, 2$, $\operatorname{grad}_h Q = (q_{\widehat{x}_1}, q_{\widehat{x}_2})^*$, $LY_2 = (\Lambda y_2, \Lambda z_2)^*$, $\Lambda v = \nu(v_{\overline{x}_1 x_1} + v_{\overline{x}_2 x_2}) - a_1 v_{\mathring{x}_1} - a_2 v_{\mathring{x}_2}$ (here $*$ stands for transposition).

Algorithm (4) is realized in two stages. On the first stage to determine the pressure $\widehat{q}$ we substitute expressions for velocity vector components $\widehat{y}_1$, $\widehat{z}_1$ from the first equation of (4) into the continuity equation. After some easy transformations we arrive at the following problem:

$$
\widehat{q}_{\widehat{x}_1 \widetilde{x}_1} + \widehat{q}_{\widehat{x}_2 \widetilde{x}_2} = F(y_2, z_2), \quad x \in S_h, \quad \widehat{q}_n = f(y_2, z_2), \quad x \in T_h,
\tag{5}
$$

where

$$
\begin{aligned}
\widehat{q}_n^{\,i-1/2,\ 1/2} &= 0.25(\widehat{q}_{x_2}^{\,i+1/2,\ 1/2} + 2\widehat{q}_{x_2}^{\,i-1/2,\ 1/2} + \widehat{q}_{x_2}^{\,i-3/2,\ 1/2}) + 0.25h_2(\widehat{q}_{\overline{x}_1 x_1}^{\,i-1/2,\ 3/2} + \widehat{q}_{\overline{x}_1 x_1}^{\,i-1/2,\ 1/2}), \\
\widehat{q}_n^{\,i-1/2,\ N_2-1/2} &= 0.25(-\widehat{q}_{\overline{x}_2}^{\,i+1/2,\ N_2-1/2} - 2\widehat{q}_{\overline{x}_2}^{\,i-1/2,\ N_2-1/2} - \widehat{q}_{\overline{x}_2}^{\,i-3/2,\ N_2-1/2}) + \\
&\quad + 0.25h_2(\widehat{q}_{\overline{x}_1 x_1}^{\,i-1/2,\ N_2-3/2} + \widehat{q}_{\overline{x}_1 x_1}^{\,i-1/2,\ N_2-1/2}), \\
\widehat{q}_n^{\,1/2,\ j-1/2} &= 0.25(\widehat{q}_{x_1}^{\,1/2,\ j+1/2} + 2\widehat{q}_{x_1}^{\,1/2,\ j-1/2} + \widehat{q}_{x_1}^{\,1/2,\ j-3/2}) + 0.25h_1(\widehat{q}_{\overline{x}_2 x_2}^{\,3/2,\ j-1/2} + \widehat{q}_{\overline{x}_2 x_2}^{\,1/2,\ j-1/2}), \\
\widehat{q}_n^{\,N_1-1/2,\ j-1/2} &= 0.25(-\widehat{q}_{\overline{x}_1}^{\,N_1-1/2,\ j+1/2} - 2\widehat{q}_{\overline{x}_1}^{\,N_1-1/2,\ j-1/2} - \widehat{q}_{\overline{x}_1}^{\,N_1-1/2,\ j-3/2}) + \\
&\quad + 0.25h_1(\widehat{q}_{\overline{x}_2 x_2}^{\,N_1-3/2,\ j-1/2} + \widehat{q}_{\overline{x}_2 x_2}^{\,N_1-1/2,\ j-1/2}),
\end{aligned}
$$

$$\widehat{q}_n^{1/2,\ 1/2} = 0.5h_2(\widehat{q}_{x_1}^{1/2,\ 3/2} + \widehat{q}_{x_1}^{1/2,\ 1/2})/(h_1+h_2) + 0.5h_1(\widehat{q}_{x_2}^{3/2,\ 1/2} + \widehat{q}_{x_2}^{1/2,\ 1/2})/(h_1+h_2),$$

$$\widehat{q}_n^{1/2,\ N_2-1/2} = 0.5h_2(\widehat{q}_{x_1}^{1/2,\ N_2-3/2} + \widehat{q}_{x_1}^{1/2,\ N_2-1/2})/(h_1+h_2) + 0.5h_1(-\widehat{q}_{\overline{x}_2}^{3/2,\ N_2-1/2} - \widehat{q}_{\overline{x}_2}^{1/2,\ N_2-1/2})/(h_1+h_2),$$

$$\widehat{q}_n^{N_1-1/2,\ N_2-1/2} = 0.5h_2(-\widehat{q}_{\overline{x}_1}^{N_1-1/2,\ N_2-3/2} - \widehat{q}_{\overline{x}_1}^{N_1-1/2,\ N_2-1/2})/(h_1+h_2) + 0.5h_1(-\widehat{q}_{\overline{x}_2}^{N_1-3/2,\ N_2-1/2} - \widehat{q}_{\overline{x}_2}^{N_1-1/2,\ N_2-1/2})/(h_1+h_2),$$

$$\widehat{q}_n^{N_1-1/2,\ 1/2} = 0.5h_2(-\widehat{q}_{\overline{x}_1}^{N_1-1/2,\ 3/2} - \widehat{q}_{\overline{x}_1}^{N_1-1/2,\ 1/2})/(h_1+h_2) + 0.5h_1(\widehat{q}_{x_2}^{N_1-3/2,\ 1/2} + \widehat{q}_{x_2}^{N_1-1/2,\ 1/2})/(h_1+h_2),$$

$$x \in T_h, \quad F(y_2,z_2) = (\Lambda y_2)_{\widetilde{x}_1} + (\Lambda z_2)_{\widetilde{x}_2}, \quad x \in S_h,$$

and the function f is directly obtained via both the continuity equation at the nodes $x \in T_h$ and the first two motion equations in (4).

Thus, after the first stage of the realization of algorithm (4) we have at the grid Ω_h the values of $\widehat{q}$ (up to an arbitrary addend), which, as we shall show later, determine on ω_h in a unique way the $\operatorname{grad}_h \widehat{Q}$. The second vector equation represents two five-point difference schemes for $\widehat{y}_2$, $\widehat{z}_2$, which are taken in the capacity of a solution approximate to the components of the velocity vector u_1, u_2. These schemes can be solved by both direct and iterative methods.

Let us study the questions of resolvability of the problem for the pressure. Consider the difference scheme

$$\Lambda y \equiv y_{\widehat{x}_1\widetilde{x}_1} + y_{\widehat{x}_2\widetilde{x}_2} = F, \quad x \in S_h, \quad y_n = f, \quad x \in T_h, \tag{6}$$

which is an analog of scheme (5). We introduce notation: U is a set of angular points of Ω_h; Π' (Π'') is a set of the nodes from given set, for which $i+j$ is even (odd); S_h' (S_h'') is a set of even (odd) nodes in S_h; T_{1h} is a set of nodes of the left and right boundary in T_h, T_{2h} is a set of nodes of upper and lower boundaries in T_h.

Lemma. *For problem* (6) *the following inequalities take place*

$$\sum_{S_h} h_1h_2F + \sum_{T_{1h}\setminus U} h_2f + \sum_{T_{2h}\setminus U} h_1f + \sum_{U} 0.5(h_1+h_2)f = 0,$$

$$\sum_{S_h'} h_1h_2F + \sum_{T_{1h}'\setminus U'} h_2f + \sum_{T_{2h}'\setminus U'} h_1f + \sum_{U'} 0.5(h_1+h_2)f = 0,$$

$$\sum_{S_h''} h_1h_2F + \sum_{T_{1h}''\setminus U''} h_2f + \sum_{T_{2h}''\setminus U''} h_1f + \sum_{U''} 0.5(h_1+h_2)f = 0.$$

The validity of Lemma can be verified directly if we replace in the equalities under consideration F and f with their expressions from (5); it is a direct consequence of the relations

$$\sum_{\Omega_h}(u_{\widetilde{x}_1} + v_{\widetilde{x}_2}) = 0, \quad \sum_{\Omega_h'}(u_{\widetilde{x}_1} + v_{\widetilde{x}_2}) = 0, \quad \sum_{\Omega_h''}(u_{\widetilde{x}_1} + v_{\widetilde{x}_2}) = 0, \quad u,\ v \in H_1.$$

Further, we write problem (6) in the operator form:

$$Ay = g, \tag{7}$$

where $A = -\overline{\Lambda}$ is a linear bounded selfadjoint operator in the finite-dimensional Hilbert space H_2 (see [20]). Let us turn to the theory of operator equations. Let $\operatorname{Ker} A$ be the kernel of the operator A, $\operatorname{Im} A$ being its image. Then the space H_2 is the direct sum of the orthogonal subspaces: $H_2 = \operatorname{Ker} A + \operatorname{Im} A^*$, $H_2 = \operatorname{Ker} A^* + \operatorname{Im} A$. As known, equation (7) is uniquely resolvable with any right-hand side $g \in H_2$ if and only if $\operatorname{Ker} A = 0$. If $\operatorname{Ker} A \neq 0$, then for the resolvability of non-homogeneous equation (7) it is necessary and sufficient that the right-hand side g be orthogonal to the subspace $\operatorname{Ker} A^*$. In this case, the solution is not unique and is determined up to an arbitrary

element belonging to $\operatorname{Ker} A$: $y = y_1 + y_2$, $y_1 \in \operatorname{Ker} A$, $Ay_2 = g$, $y_2 \in \operatorname{Im} A^*$. Thus constructed, solution y of problem (7) is classical. If, in addition, y has the least norm ($\operatorname{Ker} A \neq 0$, g is orthogonal to $\operatorname{Ker} A^*$), then this solution is unique, belongs to $\operatorname{Im} A^*$, and is said to be normal.

Let us formulate the theorem on existence of the classical solution of operator equation (7), and thus of the difference problem (5), which arises at the first stage of the realization of algorithm (4).

Theorem 1. *Lemma guarantees the existence, in the classical sense, of a solution of problem* (7).

Theorem 1 can be proved by the direct verification by means of the equalities $[g, \mu_{11}]_{H_2} = 0$ and $[g, \mu_{N_1N_2}]_{H_2} = 0$, where $\mu_{11}, \mu_{N_1N_2}$ are eigenfunctions of the operator A of problem (7), which form a basis in the subspace $\operatorname{Ker} A$ (all remaining eigenfunctions form a basis in the subspace $\operatorname{Im} A$).

Since the basis of the subspace $\operatorname{Ker} A$ consists of two functions (μ_{11} and $\mu_{N_1N_2}$) and an arbitrary element (up to which the solution of problem (7) is determined) belongs to the given subspace, this element is a certain linear combination of the functions μ_{11}, $\mu_{N_1N_2}$. It can be divided into two different constants, one being defined at the nodes of the grid Ω_h', while other at the nodes of the grid Ω_h''. However, in this situation the gradients $y_{\widehat{x}_1}$, $y_{\widehat{x}_2}$, obtained on the cells of the grid Ω_h, will be determined in a unique way. The unique normal solution $y \in \operatorname{Im} A$ can be found in a way similar to that in [20].

Thus difference scheme (4) is an example of the application of MADM for splitting with respect to the physical processes; moreover, in this case, the complete approximation of the initial problem remains same and the classical conditions of resolvability are fulfilled.

Write $\|u\|^2_{(1)} = \|u_{\overline{x}_1}]\|^2_1 + |[u_{x_1}\|^2_1 + \|u_{\overline{x}_2}]\|^2_1 + |[u_{x_2}\|^2_1$. For the stability of difference scheme (4) the following theorem takes place.

Theorem 2. *Difference scheme* (4) *is unconditionally stable with respect to initial data, for its solution the estimate takes place*

$$B(Y, Q) \le C_2(\|\varphi_1\|^2_{(1)} + \|\varphi_2\|^2_{(1)} + \|\Lambda\varphi_1\|^2 + \|\Lambda\varphi_2\|^2),$$

where $B(Y, Q) = \|y_2\|^2_{(1)} + \|z_2\|^2_{(1)} + C_1(\|\Lambda y_2 - q_{\widehat{x}_1}\|^2 + \|\Lambda z_2 - q_{\widehat{x}_2}\|^2)$, $C_1 > 0$, $C_2 > 0$ *are constants independent of grid steps.*

In the proof of Theorem 2 the technique is used applicable for obtaining similar estimates of stability of MADM (see [1]–[4]). Since difference scheme (4) satisfies the condition of complete approximation, one can easily see that with sufficient smoothness of the desired solution the scheme has the rate of convergence of order $O(h^2 + \tau)$. A worsening of the smoothness properties will intrinsically lead to slower rate of convergence of the method.

Along with difference scheme (4), for solving problem (1)–(3) one can also use an algorithm of asynchronous type:

$$\begin{gathered} Y_{1t} = -\sigma \operatorname{grad}_h(\widehat{Q} - Q) - \operatorname{grad}_h Q + LY_2, \\ Y_{2t} = \sigma L(\widehat{Y}_2 - Y_2) - \operatorname{grad}_h Q + LY_2, \quad \operatorname{div}_h \widehat{Y}_1 = 0. \end{gathered} \tag{8}$$

For the stability of difference scheme (8) the assertion of Theorem 2 remains valid for $\sigma \ge 1$. Obviously, application of scheme (8) causes an increase in potentiality for parallelizing of the computational process.

Basing on scheme (4), one can easily construct economic three-component methods, for example, those of the form:

$$\begin{gathered} Y_{1t} = -\operatorname{grad}_h \widehat{Q} + L_2Y_2 + L_3Y_3, \quad Y_{2t} = -\operatorname{grad}_h \widehat{Q} + L_2\widehat{Y}_2 + L_3Y_3, \\ Y_{3t} = -\operatorname{grad}_h \widehat{Q} + L_2\widehat{Y}_2 + L_3\widehat{Y}_3, \quad \operatorname{div}_h \widehat{Y}_1 = 0, \end{gathered} \tag{9}$$

or

$$\begin{gathered} Y_{1t} = -\sigma \operatorname{grad}_h(\widehat{Q} - Q) + M(Q, Y_2, Y_3), \\ Y_{\alpha t} = \sigma L_\alpha(\widehat{Y}_\alpha - Y_\alpha) + M(Q, Y_2, Y_3), \quad \alpha = 2, 3, \quad \operatorname{div}_h \widehat{Y}_1 = 0, \end{gathered} \tag{10}$$

where $M(Q,Y_2,Y_3) = -\operatorname{grad}_h Q + L_2Y_2 + L_3Y_3$, $L_{\alpha+1}Y_{\alpha+1} = (\Lambda_{\alpha+1}y_{\alpha+1}, \Lambda_{\alpha+1}z_{\alpha+1})^*$, $Y_\alpha = (y_\alpha, z_\alpha)^*$, $\Lambda_{\alpha+1}y_{\alpha+1} = \nu y_{\alpha+1,\,\overline{x}_\alpha x_\alpha} - a_\alpha y_{\alpha+1,\,\mathring{x}_\alpha}$, $\Lambda_{\alpha+1}z_{\alpha+1} = \nu z_{\alpha+1,\,\overline{x}_\alpha x_\alpha} - a_\alpha z_{\alpha+1,\,\mathring{x}_\alpha}$, $\alpha = 1,2$, $\operatorname{grad}_h Q = (q_{\widehat{x}_1}, q_{\widehat{x}_2})^*$.

The following theorem takes place.

Theorem 3. *Difference schemes* (9), (10) *are unconditionally stable with respect to initial data* (*scheme* (10) *is stable for* $\sigma \geq 3/2$), *and for their solution one has the estimate*

$$\sum_{\alpha=1}^{2}(\|y_{\alpha\overline{x}_\alpha}]|_\alpha^2 + \|z_{\alpha\overline{x}_\alpha}]|_\alpha^2) + C_3\Big(\|y_{3t}\|^2 + \|z_{3t}\|^2 + \Big\|\sum_{\alpha=1}^{2}\Lambda_\alpha y_\alpha - q_{\widehat{x}_1}\Big\|^2 + \Big\|\sum_{\alpha=1}^{2}\Lambda_\alpha z_\alpha - q_{\widehat{x}_2}\Big\|^2\Big) \leq$$
$$\leq C_4\Big(\sum_{\alpha=1}^{2}(\|\varphi_{1\overline{x}_\alpha}]|_\alpha^2 + \|\varphi_{2\overline{x}_\alpha}]|_\alpha^2) + \Big\|\sum_{\alpha=1}^{2}\Lambda_\alpha\varphi_1 - q_{0\widehat{x}_1}\Big\|^2 + \Big\|\sum_{\alpha=1}^{2}\Lambda_\alpha\varphi_2 - q_{0\widehat{x}_2}\Big\|^2\Big),$$

where $C_3, C_4 = \text{const} > 0$.

Methods (8), (10) have some restrictions, e. g., related to the quantity of spatial variables. For finding Q, the direct method, based on the fast Fourier transformation, is usually used. As concerns the problem of determination of the component Y_2, the spatial decomposition method has some additional possibilities in this direction. Let us consider some versions of this method in application to solving system (8) for $\sigma = 1$.

We decompose the domain $\overline{G}$ into subdomains by the straight lines x_{1i}, $i = 1,\dots,N$, which are parallel to the axis Ox_2, so that the nodes of the grid $x_{i_1} = i_1h_1$, $i_1 = 0,\dots,N_1$, lie on the corresponding straight lines. By the same token the grid $\overline{\omega}_h$ is divided into subdomains $\overline{\omega}_h^i$ (Fig. 1).

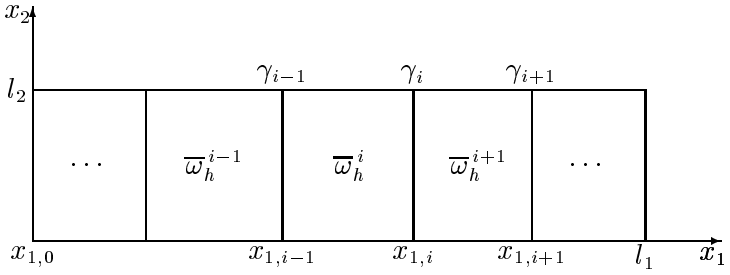

Fig. 1. A fragment of decomposition of the domain

Let $Y_2^{(i)}(x)$ be determined for $x \in \overline{\omega}_h^i$. In a subdomain $\overline{\omega}_h^i$ we define the grid operators $L_1u = -\nu h_1^{-1}u_{\overline{x}_1} + 0.5u_{\overline{x}_2x_2} + 0.5h_1^{-1}a_1u^{(-1_1)} - 0.5a_2u_{\mathring{x}_2}$, $L_2u = \nu h_1^{-1}u_{x_1} + 0.5u_{\overline{x}_2x_2} - 0.5h_1^{-1}a_1u^{(+1_1)} - 0.5a_2u_{\mathring{x}_2}$. Then the method of decomposition of domain for computing the components of the velocity can be written as follows:

$$Y_{1t} = -\operatorname{grad}_h\widehat{Q} + LY_2, \quad Y_{2t} = -\operatorname{grad}_h Q + L\widehat{Y}_2, \quad \operatorname{div}_h\widehat{Y}_1 = 0, \quad (x,t) \in \omega_h^i \times \omega_\tau,$$
$$\frac{\widehat{Y}_2^{(i)} - 0.5(Y_2^{(i-1)} + Y_2^{(i)})}{\tau} = -\operatorname{grad}_h Q + \sigma L_2(\widehat{Y}_2^{(i)} - Y_2^{(i)}) + L_1Y_2^{(i-1)} + L_2Y_2^{(i)}, \quad (x,t) \in \gamma_{i-1} \times \omega_\tau, \tag{11}$$
$$\frac{\widehat{Y}_2^{(i)} - 0.5(Y_2^{(i+1)} + Y_2^{(i)})}{\tau} = -\operatorname{grad}_h Q + \sigma L_2(\widehat{Y}_2^{(i)} - Y_2^{(i)}) + L_1Y_2^{(i)} + L_2Y_2^{(i+1)}, \quad (x,t) \in \gamma_i \times \omega_\tau.$$

For the stability of algorithm (11) we have

Theorem 4. *The method of decomposition* (11) *is unconditionally stable for* $\sigma \geq 1$, *and for estimation of its solution the inequality takes place*

$$\sum_{i=0}^{N-1}(B^{(i)}(Y,Q) + b^{(i)}) \leq C_5(\|\varphi_1\|_{(1)}^2 + \|\varphi_2\|_{(1)}^2 + \|\Lambda\varphi_1\|^2 + \|\Lambda\varphi_2\|^2),$$

where $B^{(i)}(Y,Q)$ *is the norm of* $B(Y,Q)$ *for the subdomain* $\omega_h^{(i)}$, $b_i = \|Y_2^{i+1} - Y_2^{(i)}\|^2$ *for* $x \in \gamma_i$, $C_5 = \text{const} > 0$.

If the quantity of subdomains is finite and does not depend on steps of the grid, then method (11) converges on smooth solutions with the rate $O(h^2 + \tau)$. This improves the known results on the method of decomposition, constructed on the base of MADM in [21]–[23], and also the classical two-component alternating direction method, in which the estimate of accuracy had the order $h^2 + h^{-1/2}\tau$. Decomposition into subdomains in method (11) is possible simultaneously with respect to the second variable. If the quantity of subdomains increases essentially and approaches by the order to the number of nodes of the spatial grid, then the accuracy of the method worsens, but it always has the order $h^2 + h^{-1/2}\tau$.

The analysis of the error of various economic methods for solving the Navier–Stokes equations shows that they have some disequilibriums, whose presence leads to a violation of intrinsic conservativeness of the initial problem. Therefore the usual implicit schemes, which approximate with high order of adequacy the differential problem, have an advantage (especially when one uses "rough" grids). In connection with the latter, a question on construction of efficient methods for realization of implicit schemes arises. It turns out that within the framework of the algorithm suggested above one can construct some economic iterative methods which preserve all features of the initial model and admit an adaptation for multiprocessor computers.

Consider for equation (1) a purely implicit scheme of order $O(h^2 + \tau)$:

$$Y_t = -\operatorname{grad}_h \widehat{Q} + L\widehat{Y}, \quad \operatorname{div}_h \widehat{Y} = 0, \tag{12}$$

with the natural initial and boundary conditions. To solve (12) we construct the following iterative processes:

$$\overset{k+1}{Y}_{1t} = -\operatorname{grad}_h \overset{k+1}{Q} + L\overset{k}{Y}_2, \quad \overset{k+1}{Y}_{2t} = -\operatorname{grad}_h \overset{k+1}{Q} + L\overset{k+1}{Y}_2, \quad \operatorname{div}_h \overset{k+1}{Y}_2 = 0, \tag{13}$$

$$\overset{k+1}{Y}_{1t} = -\sigma \operatorname{grad}_h(\overset{k+1}{Q} - \overset{k}{Q}) - \operatorname{grad}_h \overset{k}{Q} + L\overset{k}{Y}_2,$$

$$\overset{k+1}{Y}_{2t} = \sigma L(\overset{k+1}{Y}_2 - \overset{k}{Y}_2) - \operatorname{grad}_h \overset{k}{Q} + L\overset{k}{Y}_2, \quad \operatorname{div}_h \overset{k+1}{Y}_2 = 0, \tag{14}$$

where $\overset{k+1}{Y}_{\alpha t} = (\overset{k+1}{Y}_\alpha - Y)/\tau$. Obviously, the construction of iterative methods for the case of a large number of components of decomposition (and for the method of decomposition) repeats algorithms (9)–(11).

The convergence of iterative processes (13), (14) can be proved in a way similar to that in [24]. Let $\overset{k+1}{\rho}_\alpha = \widehat{Y} - \overset{k+1}{Y}_\alpha$, $\overset{k+1}{D} = -(\widehat{Q} - \overset{k+1}{Q})$. The following theorem is valid.

Theorem 5. *Iteration processes* (13), (14) *converge to the solution of implicit difference scheme* (12) *with the rate of a geometric progression, and for the estimate of the convergence rate the inequality holds*:

$$\overset{k+1}{Z} \le z^{k+1}\overset{0}{Z}, \tag{15}$$

where $\overset{k+1}{Z} = \|\overset{k+1}{\rho}_1\|^2_{(1)} + \|\overset{k+1}{\rho}_2\|^2_{(1)} + \|\operatorname{grad}_h \overset{k+1}{Q} - L\overset{k+1}{\rho}_2\|^2$, $z = 1/(1 + C_6 h^2 \tau^{-1})$, $C_6 > 0$ *is a bounded constant independent of both* h *and* τ.

Estimate (15) implies that the optimal rate of convergence of the method is attained with $\tau \sim h$. This condition is characteristical also for the ordinary alternating direction method. Analogous iterative processes can be constructed for the algorithm of decomposition of domain (11).

Obviously, the implicit difference scheme can be considered as an iterative method for solving a stationary problem for the Navier–Stokes equations

$$\nu\sum_{\alpha=1}^{2}\frac{\partial^2 u_k}{\partial x_\alpha^2}+\sum_{\alpha=1}^{2}a_\alpha\frac{\partial u_k}{\partial x_\alpha}=-\frac{\partial P}{\partial x_k}+f_k(x),\quad k=1,2,\quad \sum_{\alpha=1}^{2}\frac{\partial u_\alpha}{\partial x_\alpha}=0,\quad x\in G,\quad u_k(x)=0,\quad x\in\Gamma.$$

Algorithms (13), (14) in this situation must be considered as interior iterative processes. Let us note that an increase of problem's dimension does not create principal difficulties in the construction of similar algorithms, because the quantity of decompositions in the use of MADM can be arbitrary.

Similar results can be obtained for the nonlinear problem (see [3]):

$$\frac{\partial u_k}{\partial t}-\nu\Delta u_k+\sum_{\alpha=1}^{2}\frac{\partial}{\partial x_\alpha}(u_\alpha u_k)=-\frac{\partial P}{\partial x_k},\quad \sum_{\alpha=1}^{2}\frac{\partial u_\alpha}{\partial x_\alpha}=0 \tag{16}$$

with conditions (3). On the base of energetically neutral difference schemes (see [10]) for numerical solving problem (16) (and also for the stationary problem) some effective economic iterative methods can be constructed, which are similar to those cited above.

As we have noted above, one of the merits of MADM consists of the following. In using this approach in the problems on decomposition of domain, practically, no restrictions on the quantity of subdomains arise. Owing to this property, one can construct algorithms of spatial decomposition on the base of MADM. In these algorithms subdomains of decomposition are identified with elementary cells of the grid. The present approach seems to be rather perspective in its use in the capacity of an iterative process of realization of implicit difference schemes and in solving stationary problems by the method of establishment. We consider a class of iterative algorithms based on cell-by-cell decomposition of the problem.

In order to sew together the difference solutions at each four vertices of cell grid we use additive representation of the difference operator of the problem. In doing so, in contrast to the traditional alternating direction method we use not the coordinate-wise splitting of the operator, but splitting with respect to diagonal directions. In dependence on the choice of a diagonal direction two variants of such splitting exist $L=L_m+L_{m+2}$, $m=1,2$,

$$\begin{aligned}
L_1u&=\nu(h_1^{-1}u_{x_1}+h_2^{-1}u_{x_2})-0.5(h_1^{-1}a_1u^{(+1_1)}+h_2^{-1}a_2u^{(+1_2)}),\\
L_3u&=-\nu(h_1^{-1}u_{\overline{x}_1}+h_2^{-1}u_{\overline{x}_2})+0.5(h_1^{-1}a_1u^{(-1_1)}+h_2^{-1}a_2u^{(-1_2)}),\\
L_2u&=-\nu(h_1^{-1}u_{\overline{x}_1}-h_2^{-1}u_{x_2})+0.5(h_1^{-1}a_1u^{(-1_1)}-h_2^{-1}a_2u^{(+1_2)}),\\
L_4u&=\nu(h_1^{-1}u_{x_1}-h_2^{-1}u_{\overline{x}_2})-0.5(h_1^{-1}a_1u^{(+1_1)}-h_2^{-1}a_2u^{(-1_2)}).
\end{aligned}$$

By numerating the nodes of cells counterclockwise, starting with the left lowest, for the set of nodes of a cell we get a system of equations for computation of velocities on the $(k+1)$-st iteration

$$\frac{\overset{k+1}{Y_2}{}^{(i)}-0.5(\widetilde{Y}_2^{(i+2)}+Y_2^{(i)})}{\tau}=-\operatorname{grad}_h Q+\sigma L_i\big(\overset{k+1}{Y_2}{}^{(i)}-\overset{k}{Y_2}{}^{(i)}\big)+L_i\overset{k}{Y_2}{}^{(i)}+L_{i+2}\overset{k}{\widetilde{Y}_2}{}^{(i+2)},\tag{17}$$

$i=1,\dots,4$, $L_{i+4}\equiv L_i$, $Y_2^{(i+4)}\equiv Y_2^{(i)}$. Here $\widetilde{Y}_2^{(i+2)}$ are solutions of the problems in subdomains which are diagonally adjacent to the cell under consideration in the grid nodes, which have ordering numbers $i=1,\dots,4$. One can easily note that, in an analogous manner of numeration of nodes, to each interior node two difference solutions will be related, belonging to two diagonally adjacent to it cells $Y_2^{(i)}$, $\widetilde{Y}_2^{(i+2)}$.

The number of equations in (17) coincides with the quantity of unknown grid functions on the $(k+1)$-st iteration. By virtue of the latter, the algorithm becomes a closed loop on a separate cell, which enables us to realize computation at each of the subdomains independently of other ones. Thus, we have constructed an iterative algorithm of explicit type of realization with maximal possibilities for its parallelizing.

In the use of iterative algorithm (17) for finding a stationary solution of the problem by the method of establishment, the rate of convergence of the algorithm in the whole (the sum of iterations in (17) with respect to all temporary layers) practically does not depend on the accuracy of the interior iteration in (17), which is attained on intermediary layers. This allows at each evolution step to restrict ourselves to one iteration, which leads to the iterative algorithm of the ripple–through computing.

In the conclusion, let us briefly discuss the estimate of computational qualities of the multicomponent iterative algorithms which are based on cell-by-cell decomposition of the Navier–Stokes problem. We think that the most essential point here is to clarify the dependence of the rate of convergence of iterations on the steps of spatial grid, the values of iteration parameters, and the Reynolds number. It is considered rather satisfactory if in order to attain a prescribed accuracy in computation of stationary picture of two-dimensional viscous flows the number of iterations grows not faster than $O(\nu^{-1})$. A computational experiment for a model problem of convection–diffusion shows the following (see [25]): For the class of methods of the decomposition of the type (17) the number of iterations for obtaining a stationary solution with a given accuracy is estimated as $O(\nu^{-1/2})$. In addition, the numerical verification shows that the best rate of convergence of those iterative methods is attained for the values of iteration parameters $\sigma = 1$ and $\tau \sim h$ (see [25]), which coincides with the estimate in Theorem 5.

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