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Variational Inequalities with Strong Nonlinearities

V. S. Klimov[1*]

[1] *Yaroslavl State University, ul. Sovetskaya 14, Yaroslavl, 150000 Russia*

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Abstract—We study a connection between critical values and topological characteristics of non-smooth functionals. We establish analogs of theorems about regular interval and "nek". We also find lower estimates of solutions to variational inequalities with odd potential operators.

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Introduction. The paper studies applications of monotone type operator theory to estimates of number of solutions of variational inequalities. The main objective is to consider the variational and boundary-value problems with nonpower nonlinearities. This character of nonlinearities generates a number of difficulties: The domain spaces of the functionals are nonseparable and nonreflexive; the functional themselves are not properly defined nondifferentiable and discontinuous in their natural domain; their derivatives are unbounded maps etc.

The clarifying example is the question on the number of generalized solutions $\mathcal{N}(\lambda)$ of the boundary-value problem

$$-\sum_{k=1}^{m} \frac{\partial}{\partial x_k}\left(2\frac{\partial u}{\partial x_k} e^{|\nabla u|^2}\right) + \lambda \Psi'(u) = 0 \quad (x \in \Omega), \quad u(x) = 0 \quad (x \in \partial\Omega).$$

Here $\Psi : \mathbb{R} \to \mathbb{R}$ is a smooth even function, $\Psi(0) = 0$, $\partial\Omega$ is the boundary of the bounded domain $\Omega \subset \mathbb{R}^m$. The results of the paper imply that if the function Ψ meets relations

$$\Psi(t) > 0 \quad (0 < t < R_0), \quad \lim_{t\to\infty} e^{-t^2}\Psi(kt) = 0 \ \forall k > 0,$$

then $\mathcal{N}(\lambda) \to \infty$ as $\lambda \to -\infty$. This statement holds true even in the case of $\Psi(t) = t^2$.

Solutions to this problem coincide with the extremal points of the functional

$$f(u) = \int_\Omega \left[e^{|\nabla u(x)|^2} + \lambda \Psi(u(x))\right] dx.$$

In order to solve this problem it seems natural to apply usual variational calculus methods. Nevertheless the nonpower nonlinearities character forces us to consider the functional f on the nonreflexive and nonseparable Orlicz—Sobolev space generated by the N-function $\Phi(t) = e^{t^2} - 1$; the functional f then becomes a nondiffferentiable one, f does not meet Palais—Smale compactness condition etc. Thus, the known variational calculus methods do not allow us to obtain a concise estimate of this functional extremal points number.

The paper is based on the ideas and methods of nonlinear analysis [1—6]. Some similar questions can be found in [7—10].

We assume the following notation: $\overline{\mathfrak{M}}$ ($\overset{\circ}{\mathfrak{M}}$, $\partial\mathfrak{M}$) are closure, interior, and boundary of some subset $\mathfrak{M}$ of the topological space $\mathfrak{R}$; $\mathrm{Pv}(L)$ is a collection of all nonempty convex subsets of a linear space L; X^* is the conjugate to the Banach space X linear functionals space; $\mathrm{Cv}\,(X)$ $(\mathrm{Kv}\,(X))$ is the part of $\mathrm{Pv}(X)$

*E-mail: vsk76@list.ru.